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Modification of the Exterior and Interior Solution of Einstein's G_{22} Field Equation for a Homogeneous Spherical Massive Bodies whose Fields Differ in Radial Size, Polar Angle, and Time.

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U. Rilwan[1*], A.U. Maisalatee[2], E. I. Ugwu[1], O. G. Okara[3], S. Muhammad[1], A. Ubaidullah[4] and H. Abdulrahman[5]

[1]Department of Physics, Nigerian Army University, P.M.B 1500 Biu, Borno State, Nigeria

[2]Liyu Unity Science Academy, Campus Avenue, Behind Yaro Sule Filling Station, P.M.B 03 Keffi, Nasarawa State.

[3]Department of Physics, Nasarawa State University, Keffi, P.M.B 1022, Nigeria

[4]Federal University Dutsin-ma, P.M.B 5001 Dutsin-ma, Katsina State, Nigeria.

[5]Department of Chemistry, Nigerian Army University, PMB 1500 Biu, Borno State, Nigeria

***Correspondence author**
U. Rilwan
Department of Physics
Nigerian Army University
P.M.B 1500 Biu, Borno State
Nigeria

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Abstract
In general theory of relativity, Einstein's field equations relate the geometry of space-time with the distribution of matter within it. These equations were first published by Einstein in the form of a tensor equation which related the local space-time curvature with the local energy and momentum within this space-time. In this article, Einstein's geometrical field equations interior and exterior to astrophysically real or hypothetical distribution of mass within a spherical geometry were constructed and solved for field whose gravitational potential varies with time, radial distance and polar angle. The exterior solution was obtained using power series. The metric tensors and the solution of the Einstein's exterior field equations used in this work has only one arbitrary function $f(t,r,\theta)$, and thus put the Einstein's geometrical theory of gravitation on the same bases with the Newton's dynamical theory of gravitation. The gravitational scalar potential $f(t,r,\theta)$ obtained in this research work to the order of $c^{o,}$ c^{-2} , contains Newton dynamical gravitational scalar potential and post Newtonian additional terms much importance as it can be applied to the study of rotating bodies such as stars. The interior solution was obtained using weak field and slow-motion approximation. The obtained result converges to Newton's dynamical scalar potential with additional time factor not found in the well-known Newton's dynamical theory of gravitation which is a profound discovery with the dependency on three arbitrary functions. Our result obeyed the equivalence principle of Physics.

Keywords: Einstein's field equation, radial distance, polar angle, Schwarzchild's metric, gravitation

Introduction
Gravitation is a natural phenomenon whose study gives a better understanding of the universe. On earth, it gives weight to physical objects and causes the ocean tides. The gravitational attraction of the original gaseous matter present in the Universe caused it to begin coalescing, forming stars and the stars to group together into galaxies so gravity is responsible for many of the large-scale structures in the Universe [1].

After the publication of Einstein's geometrical gravitational field equations (EGGFE) in 1915, the search for their exact and analytical solutions for all the gravitational fields in nature began [2-4]. Schwarzchild first constructed the exact solution to this field equation in static and pure radial spherical polar coordinates in 1916 by considering astrophysical bodies such as the sun and the stars [5] In Schwarzchild's metric, the tensor field varies with radial distance only.

A new method and approach was introduced to formulate exact analytical solutions [6] as an extension of Schwarzschild's method. This new approach took into consideration the fact that tensor field of astrophysical bodies does not depend on radial distance only as indicated in Schwarzschild's equation. This new approach was used in several studies of Einstein's geometrical field equations such as [3-8]. This method would help in the study of Ceres, Pluto, Makemake, Haumea, The Ouort Cloud and other astrosphysical bodies. In this research work, we show how exact analytical solution of the exterior and internal field equations can be constructed in the limit of c^{-2} in a gravitational field for time varying spherical massive bodies using the new method and approach.

Construction of the Exterior G_{22} Field Equation
To construct the G_{22} field equation, we applied the covariant metric tensors for this distribution of mass or pressure in

spherical polar coordinates *f(t,r,θ)* constructed by [5-10] because it is also a time varying metric tensor which depends on radial distance and polar angle which is given as

$$g_{00}=\left[1+\frac{2f(t,r,\theta)}{c^2}\right]. \qquad (2.1)$$

$$g_{11}=-\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}. \qquad (2.2)$$

$$g_{22}=-r^2 \qquad (2.3)$$

$$g_{33}=-r^2\sin^2\theta \qquad (2.4)$$

$$g_{uv}=0 \qquad (2.5)$$

where *f(t,r,θ)* is an arbitrary function, determined by the mass or pressure and possess symmetries of the latter. In approximate gravitational field, it is equal to Newton's gravitational scalar potential exterior to the spherical mass distribution.

To obtain the corresponding contravariant metric tensors for this gravitational field, the Quotient Theorem [5] of the tensor analysis was used to obtain the components of the contravariant tensor as

$$g^{00}=\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}. \qquad (2.6)$$

$$g^{11}=-\left[1+\frac{2f(t,r,\theta)}{c^2}\right]. \qquad (2.7)$$

$$g^{22}=-\frac{1}{r^2}. \qquad (2.8)$$

$$g^{33}=-\frac{1}{r^2\sin^2\theta}. \qquad (2.9)$$

$$g^{\mu\nu}=0.\ \text{otherwise} \qquad (2.10)$$

The coefficients of affine connections, defined by the metric tensors of space-time are determined [5-12] using equations (2.1)-(2.10),

$$\Gamma^{\mu}{}_{\alpha\beta}=\frac{1}{2}g^{\mu\xi}\left(g_{\alpha\xi,\beta}+g_{\beta\xi,\alpha}-g_{\alpha\beta,\xi}\right). \qquad (2.11)$$

They are found to be given explicitly in terms of (ct,r,θ)within this regionas

$$\Gamma^{0}{}_{00}=\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial t}. \qquad (2.12)$$

$$\Gamma^{0}{}_{01}=\Gamma^{0}{}_{10}=\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial r}. \qquad (2.13)$$

$$\Gamma^{0}{}_{11}=-\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-3}\frac{\partial f(t,r,\theta)}{\partial t}. \qquad (2.14)$$

$$\Gamma^{0}{}_{02}=\Gamma^{0}{}_{20}=\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}. \qquad (2.15)$$

$$\Gamma_{00}=\frac{}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]\frac{\partial f(t,r,\theta)}{\partial}. \qquad (2.16)$$

$$\Gamma^{1}{}_{01}=\Gamma^{1}{}_{10}=-\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial t}. \qquad (2.17)$$

$$\Gamma^{1}{}_{11}=-\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial r}. \qquad (2.18)$$

$$\Gamma^{1}{}_{12}=\Gamma^{1}{}_{21}=-\frac{1}{c^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-1}\frac{\partial f(t,r,\theta)}{\partial\theta}. \qquad (2.19)$$

$$\Gamma^{1}{}_{22}=-r\left[1+\frac{2f(t,r,\theta)}{c^2}\right]. \qquad (2.20)$$

$$\Gamma^{1}{}_{33}=-r\sin^2\theta\left[1+\frac{2f(t,r,\theta)}{c^2}\right]. \qquad (2.21)$$

$$\Gamma^{2}{}_{00}=\frac{1}{c^2r^2}\frac{\partial f(t,r,\theta)}{\partial\theta}. \qquad (2.22)$$

$$\Gamma^{2}{}_{11}=\frac{1}{c^2r^2}\left[1+\frac{2f(t,r,\theta)}{c^2}\right]^{-2}\frac{\partial f(t,r,\theta)}{\partial\theta}. \qquad (2.23)$$

$$\Gamma^{2}{}_{12}=\Gamma^{2}{}_{21}=\frac{1}{r}. \qquad (2.24)$$

$$\Gamma^{2}{}_{33}=-\sin\theta\cos\theta. \qquad (2.25)$$

$$\Gamma^{3}{}_{13}=\Gamma^{3}{}_{31}=\frac{1}{r}. \qquad (2.26)$$

$$\Gamma^{3}{}_{23}=\Gamma^{3}{}_{32}=\cot\theta. \qquad (2.27)$$

$$\Gamma^{\mu}{}_{\alpha\beta}=0.\ \text{; otherwise} \qquad (2.28)$$

The exterior field equation in this field is given as

$$G_{22}=R_{22}-\frac{1}{2}Rg_{22}=0. \qquad (2.29)$$

The choice of this component is because it's observed that all the solution to the field equation towards the exterior converges at the same way.
The expression for the Ricci tensor R_{22} and the curvature scalar R in this field are given respectively as:

$$R_{22}=R^{0}_{220}+R^{1}_{221}+R^{2}_{222}+R^{3}_{223}. \qquad (2.30)$$

$$R=g^{00}R_{00}+g^{11}R_{11}+g^{22}R_{22}+g^{33}R_{33}. \qquad (2.31)$$

The expanded form of equations (2.30) and (2.31) are given as

$$R_{22}=\Gamma^{0}{}_{20,2}+\Gamma^{0}{}_{20}\Gamma^{0}{}_{02}-\Gamma^{1}{}_{22}\Gamma^{0}{}_{10}+\Gamma^{1}{}_{21,2}-\Gamma^{1}{}_{22,1}+\Gamma^{1}{}_{21}\Gamma^{1}{}_{12}-\Gamma^{1}{}_{22}\Gamma^{1}{}_{11}-\Gamma^{2}{}_{12}\Gamma^{1}{}_{22}$$
$$+\Gamma^{2}_{22,2}-\Gamma^{2}_{22,2}+\Gamma^{0}_{22}\Gamma^{2}_{02}-\Gamma^{0}_{22}\Gamma^{2}_{02}+\Gamma^{1}_{22}\Gamma^{2}_{12}-\Gamma^{1}_{22}\Gamma^{2}_{12}+\Gamma^{2}_{22}\Gamma^{2}_{22}-\Gamma^{2}_{22}\Gamma^{2}_{22}+\Gamma^{3}_{22}\Gamma^{2}_{32}-\Gamma^{3}_{22}\Gamma^{2}_{32}$$
$$+\Gamma^{3}{}_{23,2}-\Gamma^{1}{}_{22}\Gamma^{3}{}_{13}+\Gamma^{3}{}_{23}\Gamma^{3}{}_{32}. \qquad (2.32)$$

$$R=g^{00}\{\Gamma^{0}_{00,0}-\Gamma^{0}_{00,0}+\Gamma^{0}_{00}\Gamma^{0}_{00}-\Gamma^{0}_{00}\Gamma^{0}_{00}+\Gamma^{1}_{00}\Gamma^{0}_{10}\Gamma^{1}{}_{01,0}-\Gamma^{1}_{00}\Gamma^{0}_{10}$$
$$-\Gamma^{2}_{00}\Gamma^{0}_{20}-\Gamma^{2}_{00}\Gamma^{0}_{20}+\Gamma^{3}_{00}\Gamma^{0}_{30}-\Gamma^{3}_{00}\Gamma^{0}_{30}-\Gamma^{1}{}_{00,1}+\Gamma^{0}{}_{01}\Gamma^{1}{}_{00}-\Gamma^{0}{}_{00}\Gamma^{1}{}_{01}+\Gamma^{1}{}_{01}\Gamma^{1}{}_{10}$$
$$-\Gamma^{1}{}_{00}\Gamma^{1}{}_{11}-\Gamma^{2}{}_{00}\Gamma^{1}{}_{21}-\Gamma^{2}{}_{00,2}+\Gamma^{0}{}_{02}\Gamma^{2}{}_{00}-\Gamma^{1}{}_{00}\Gamma^{2}{}_{12}-\Gamma^{1}{}_{00}\Gamma^{3}{}_{13}-\Gamma^{2}{}_{00}\Gamma^{3}{}_{23}\}$$
$$+g^{11}\{\Gamma^{0}{}_{10,1}-\Gamma^{0}{}_{11,0}+\Gamma^{0}{}_{10}\Gamma^{0}{}_{01}-\Gamma^{0}{}_{11}\Gamma^{0}{}_{00}+\Gamma^{1}{}_{10}\Gamma^{0}{}_{11}-\Gamma^{1}{}_{11}\Gamma^{0}{}_{10}$$

$$-\Gamma^2{}_{11}\Gamma^0{}_{20}+\Gamma^1_{11,1}-\Gamma^1_{11,1}+\Gamma^0_{11}\Gamma^1_{01}-\Gamma^0_{11}\Gamma^1_{01}+\Gamma^1_{11}\Gamma^1_{11}-\Gamma^1_{11}\Gamma^1_{11}+\Gamma^2_{11}\Gamma^1_{21}-\Gamma^2_{11}\Gamma^1_{21}$$
$$+\Gamma^3_{11}\Gamma^1_{31}-\Gamma^3_{11}\Gamma^1_{31}+\Gamma^2{}_{12,1}-\Gamma^2{}_{11,2}+\Gamma^1{}_{12}\Gamma^2{}_{11}-\Gamma^1{}_{11}\Gamma^2{}_{12}+\Gamma^2{}_{12}\Gamma^2{}_{21}+\Gamma^3{}_{13,1}-\Gamma^1{}_{11}\Gamma^3{}_{13}-\Gamma^2{}_{11}\Gamma^3{}_{23}+\Gamma^3{}_{13}\Gamma^3{}_{31}\}$$
$$+g^{22}\{\Gamma^0{}_{20,2}+\Gamma^0{}_{20}\Gamma^0{}_{02}-\Gamma^1{}_{22}\Gamma^0{}_{10}+\Gamma^1{}_{21,2}-\Gamma^1{}_{22,1}+\Gamma^1{}_{21}\Gamma^1{}_{12}-\Gamma^1{}_{22}\Gamma^1{}_{11}-\Gamma^2{}_{12}\Gamma^1{}_{22}$$
$$+\Gamma^2_{22,2}-\Gamma^2_{22,2}+\Gamma^0_{22}\Gamma^2_{02}-\Gamma^0_{22}\Gamma^2_{02}+\Gamma^1_{22}\Gamma^2_{12}-\Gamma^1_{22}\Gamma^2_{12}+\Gamma^2_{22}\Gamma^2_{22}-\Gamma^2_{22}\Gamma^2_{22}+\Gamma^3_{22}\Gamma^2_{32}-\Gamma^3_{22}\Gamma^2_{32}$$
$$+\Gamma^3{}_{23,2}-\Gamma^1{}_{22}\Gamma^3{}_{13}+\Gamma^3{}_{23}\Gamma^3{}_{32}\}+g^{33}\{-\Gamma^1{}_{00}\Gamma^3{}_{13}-\Gamma^2{}_{00}\Gamma^3{}_{23}-\Gamma^1{}_{33,1}-\Gamma^1{}_{33}\Gamma^1{}_{11}-\Gamma^2{}_{33}\Gamma^1{}_{21}$$
$$+\Gamma^3{}_{31}\Gamma^1{}_{33}-\Gamma^2{}_{33,2}-\Gamma^1{}_{33}\Gamma^2{}_{12}+\Gamma^3{}_{32}\Gamma^2{}_{33}+\Gamma^3_{33,3}-\Gamma^3_{33,3}+\Gamma^0_{33}\Gamma^3_{03}-\Gamma^0_{33}\Gamma^3_{03}$$
$$+\Gamma^1_{33}\Gamma^3_{13}-\Gamma^1_{33}\Gamma^3_{13}+\Gamma^2_{33}\Gamma^3_{23}-\Gamma^2_{33}\Gamma^3_{23}+\Gamma^3_{33}\Gamma^3_{33}-\Gamma^3_{33}\Gamma^3_{33}\}. \quad (2.33)$$

Explicitly equations (2.32) and (2.33) are given as

$$R_{22}=\frac{2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{2f(t,r,\theta)}{c^2}. \quad (2.34)$$

$$R=\frac{8}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{8}{c^2r}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)$$
$$-\frac{2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)-\frac{2}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{4f(t,r,\theta)}{c^2r^2}. \quad (2.35)$$

Substituting equations (2.3), (2.34) and (2.35) into (2.29) gives

$$G_{22}=\frac{2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{2f(t,r,\theta)}{c^2}$$
$$+\frac{1}{2}r^2\left[\frac{8}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{8}{c^2r}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\right.$$
$$\left.-\frac{2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)-\frac{2}{c^4r^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{4f(t,r,\theta)}{c^2r^2}\right]=0. \quad (2.36)$$

Simplifying equation (2.36) gives

$$G_{22}=-\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}+\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{4r^2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2$$
$$-\frac{r^2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{r^2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)=0. \quad (2.37)$$

Rearranging equation (2.37), and multiplying through with a negative sign gives

$$\frac{r^2}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)+\frac{2r}{c^2}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$
$$-\frac{4r^2}{c^4}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-3}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{r^2}{c^2}\left(1+\frac{2f(t,r,\theta)}{c^2}\right)^{-2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)=0. \quad (2.38)$$

Simplifying equation (2.38) and rearranging further gives

$$\left[r^2\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)+2r\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{c^2}\left(1-\frac{4f(t,r,\theta)}{c^2}+\ldots\right)\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2\right.$$
$$\left.-\frac{4r^2}{c^2}\left(1-\frac{6f(t,r,\theta)}{c^2}+\ldots\right)\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+r^2\left(1-\frac{4f(t,r,\theta)}{c^2}+\ldots\right)\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\right]=0. \quad (2.39)$$

$$\left[\left(\frac{\partial^2 f(t,r,\theta)}{\partial r^2}\right)+\frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r}+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{4f(t,r,\theta)}{c^4r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2\right.$$
$$\left.-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{24f(t,r,\theta)}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\right]=0.$$

(2.40)

$$\nabla^2 f(t,r,\theta)+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{4f(t,r,\theta)}{c^4r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$
$$-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{24f(t,r,\theta)}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\Bigg]=0. \quad (2.41)$$

$$\nabla^2 f(t,r,\theta)+\frac{\partial^2 f(t,r,\theta)}{\partial t^2}=0. \quad (2.42)$$

It has been shown that in the limit of   the wave equation (2.41) in the limit of weak fields reduced to:

$$\nabla^2 f(t,r,\theta)+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$
$$-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\Bigg]=0. \quad (2.43)$$

Equation (2.43) can also be written as

$$\frac{\partial^2 f(t,r,\theta)}{\partial r^2}+\frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r}-\frac{1}{r^2c^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)$$
$$-\frac{4}{c^2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2+\frac{\partial^2 f(t,r,\theta)}{\partial t^2}=0. \quad (2.44)$$

Let us now seek a solution of equation (2.44) in the form

$$f(t,r,\theta)=\sum_{n=0}^{\infty}R_n(r)\exp n\left(t-\frac{r\theta}{c}\right). \quad (2.45)$$

where $R_n(r)$ functions of r only. By obtaining the first and second derivatives partially of equation (2.45) for *f(t,r,θ)*; it can be shown trivially that the separate terms of the expanded equation can be shown in the equations below:

$$\frac{\partial^2 f(t,r,\theta)}{\partial r^2}=R_0{}^{11}(r)+\left(R_1{}^{11}(r)-\frac{2\theta}{c}R_1{}^1(r)-\frac{\theta^2}{c^2}R_1(r)\right)\exp\left(t-\frac{r\theta}{c}\right)$$
$$+\left(R_2{}^{11}(r)-\frac{2.2\theta}{c}R_2{}^1(r)-\frac{2^2\theta^2}{c^2}R_2(r)\right)\exp 2\left(t-\frac{r\theta}{c}\right)+\left(R_3{}^{11}(r)-\frac{2.3\theta}{c}R_3{}^1(r)-\frac{3^2\theta^2}{c^2}R_3(r)\right) \quad (2.46)$$

$$\frac{2}{r}\frac{\partial f(t,r,\theta)}{\partial r}=\frac{2}{r}R_0{}^1(r)+\frac{2}{r}R_1{}^1(r)\exp\left(t-\frac{r\theta}{c}\right)+\frac{2}{r}R_2{}^1(r)\exp 2\left(t-\frac{r\theta}{c}\right)+\frac{2}{r}R_3{}^1(r)\exp 3\left(t-\frac{r\theta}{c}\right)$$
$$-\frac{2\theta}{cr}R_1(r)\exp\left(t-\frac{r\theta}{c}\right)-\frac{2.2\theta}{cr}R_2(r)\exp 2\left(t-\frac{r\theta}{c}\right)-\frac{2.3\theta}{cr}R_3(r)\exp 3\left(t-\frac{r\theta}{c}\right)+\ldots \quad (2.47)$$

$$\frac{1}{r^2c^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2=\frac{1}{c^4}R_1{}^2(r)\exp 2\left(t-\frac{r\theta}{c}\right)+\frac{2^2}{c^4}R_2{}^2(r)\exp 4\left(t-\frac{r\theta}{c}\right)+\frac{3^2}{c^4}R_3{}^2(r)\exp 6\left(t-\frac{r\theta}{c}\right)+\ldots \quad (2.48)$$

$$\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)=\frac{4}{c^2}R_0(r)R_1(r)\exp\left(t-\frac{r\theta}{c}\right)+\frac{4.2^2}{c^2}R_0(r)R_2(r)\exp 2\left(t-\frac{r\theta}{c}\right)$$
$$+\frac{4.3^2}{c^2}R_0(r)R_3(r)\exp 3\left(t-\frac{r\theta}{c}\right)+\frac{4}{c^2}R_1{}^2(r)\exp 2\left(t-\frac{r\theta}{c}\right)+\frac{20}{c^2}R_1(r)R_2(r)\exp 3\left(t-\frac{r\theta}{c}\right)$$
$$+\frac{40}{c^2}R_1(r)R_3(r)\exp 4\left(t-\frac{r\theta}{c}\right)+\frac{4.2^2}{c^2}R_2{}^2(r)\exp 4\left(t-\frac{r\theta}{c}\right)+\frac{52}{c^2}R_2(r)R_3(r)\exp 6\left(t-\frac{r\theta}{c}\right)+\ldots \quad (2.49)$$

$$\frac{4}{c^2}\left(\frac{\partial f(t,r,\theta)}{\partial t}\right)^2 = \frac{4}{c^2}R_1^{\,2}(r)\exp 2\left(t-\frac{r\theta}{c}\right)+\frac{4.2^2}{c^2}R_2^{\,2}(r)\exp 4\left(t-\frac{r\theta}{c}\right)+\frac{4.3^2}{c^2}R_3^{\,2}(r)\exp 6\left(t-\frac{r\theta}{c}\right)+\ldots \tag{2.50}$$

$$\frac{\partial^2 f(t,r,\theta)}{\partial t^2} = R_1(r)\exp\left(t-\frac{r\theta}{c}\right)+2^2R_2(r)\exp 2\left(t-\frac{r\theta}{c}\right)+3^2R_3(r)\exp 3\left(t-\frac{r\theta}{c}\right)+\ldots \tag{2.51}$$

Comparing coefficients of

$$R_0^{\,11}(r)+\frac{2}{r}R_0^{\,1}(r)=0. \tag{2.52}$$

Solving equation (2.52) to obtain the auxiliary solution for the second order partial differential equation gives:

$$R_0(r)=-\frac{2}{r}. \tag{2.53}$$

But according to Newton's dynamical theory, Newton's gravitational scalar potential exterior to a distribution of mass or pressure is given by

$$f(r)=-\frac{GM_0}{r}. \tag{2.54}$$

Comparing equation (2.53) with Newton's gravitational scalar potential (2.54) we can choose the most convenient astrophysical solution for (2.52) as shown below:

$$R_0(r)\approx-\frac{k}{r}. \tag{2.55}$$

Where $k = GM_0$; deducing from Schwarzschild's metric and Newton's dynamical theory of gravitation, G is the universal gravitational constant and M_0 is the total mass of the spherical body.

Equating coefficients of $\exp\left(t-\frac{r\theta}{c}\right)$ yields

$$R_1^{\,11}(r)+2\left[\frac{1}{r}-\frac{\theta}{c}\right]R_1^{\,1}(r)-\frac{\theta}{c}\left[\theta+\frac{2}{r}+\frac{4}{c}R_0(r)\right]R_1(r)=0.$$

(2.56)

This is the exact differential equation for R_1 and it determines R_1 in terms of R_0. Thus, the solution assumes an exact wave equation, which in the order of C^o reduces to

$$f(t,r,\theta)\approx-\frac{k}{r}\exp\left(t-\frac{r\theta}{c}\right). \tag{2.57}$$

Construction of the Interior G_{22} Field Equation

Einstein's field equation interior to a homogeneous spherical distribution of mass is given generally as [6-16]

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{8\pi GT_{\mu\nu}}{c^4}. \tag{2.58}$$

Where the speed of light in vacuum is, $T_{\mu\nu}$ is the energy-momentum tensor due to any distribution of mass or pressure and G is the universal gravitational constant.

Now, let us assume that the homogeneous mass distribution is in a weak field limit. We can neglect the contribution from the source, thus define the energy-momentum tensor given as

$$T_{\mu\nu}=\frac{1}{2}\rho_o c^2. \tag{2.59}$$

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{4\pi G\rho_o}{c^2}. \tag{2.60}$$

Where ρ_o is the density
c is the speed of light in vacuum.
It was observed in [13] that the exterior field equations along the G_{00}, G_{22} and G_{33} converge within the exterior field, similarly along the interior field.
For mathematical convenience, we choose G_{22}
Hence the non-trivial field equation is

$$R_{22}-\frac{1}{2}Rg_{22}=\frac{4\pi G\rho_o}{c^2}. \tag{2.61}$$

Substituting the left hand side of equation (2.41) into equation (2.61) gives

$$\nabla^2 f(t,r,\theta)+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2+\frac{4f(t,r,\theta)}{c^4r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$
$$-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2+\frac{24f(t,r,\theta)}{c^4}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\Bigg]=\frac{4\pi G\rho_o}{c^2}. \tag{2.62}$$

$$\nabla^2 f(t,r,\theta)+\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)-\frac{1}{c^2r^2}\left(\frac{\partial f(t,r,\theta)}{\partial\theta}\right)^2$$
$$-\frac{4}{c^2}\left[\frac{\partial f(t,r,\theta)}{\partial t}\right]^2-\frac{4f(t,r,\theta)}{c^2}\left(\frac{\partial^2 f(t,r,\theta)}{\partial t^2}\right)\Bigg]=\frac{4\pi G\rho_o}{c^2}. \tag{2.63}$$

To the weak field limit of co, the equation reduces to

$$\nabla^2 f(t,r,\theta)+\frac{\partial^2 f(t,r,\theta)}{\partial t^2}=\frac{4\pi G\rho_o}{c^2}. \tag{2.64}$$

Conclusion

The result obtained in (2.64) is the Newton dynamical scalar field equation with an additional time factor which signifies the dynamical nature of the system, it is indeed a profound discovery, it confirms our assumption in [8] that Newton dynamical theory of gravitation NDTG is a limiting case of Einstein's geometrical gravitational field equations EGGFE, and this should clear the objection 7, 38,41 in [17]. Experimentally established equivalence principle of physics is shown with the dependency of the scalar function on time, radial distance and polar angle.

If the pressure is negligible compared to mass density [6], hence

ρ_0

Our scalar potential will be

$$\nabla^2 f(t,r,\theta)+\frac{\partial^2 f(t,r,\theta)}{\partial t^2}=-\frac{GM_0}{r} \tag{2.65}$$

Where $GM_0 = k$

The EFE reduces to Newton's law of gravitation by using both weak field approximation and slow motion approximation, eqn. (2.60) will thus further splits to four non-linear equations analogous to Maxwell's equation and could thus be applied in the study of the Gravitoelectric and Gravitomagnetic coupling phenomena.

Interestingly and remarkably, we obtain an arbitrary function equation (2.57) which is a function of radial distance, polar angle and time equal to Newton's scalar potential, hence our obtained result could be apply in all the applications of Newton's scalar potential with a much wider application such as the study of coupling effects of electromagnetism, weak field approximation.

The result obtained in equation (2.57) is applicable to all 2-D dynamical physical systems rotating about a fixed point or a phenomenon originating from a fixed point [18].

Furthermore the obtained result equation (2.57) differ from [7,6,8] in the sense that [7] is for a hypothetical systems which varies with azimuthal angle only, whereas [3] is for static homogenous oblate spheroidal systems, and [8] is for a static astrophysical systems which varies with radial distance and azimuthal angle only.

Instructively, our single dependent function $f(t,r,\theta)$ which is our physically and mathematically most satisfactory solution contains unknown post Newtonian terms or pure Einsteinian gravitational terms in order of c^{o} and c^{-2} . Hence, this research work has shown that the Exterior EGGFE can be obtained as a generalization or completion of Newton's dynamical gravitational field equations.

Interestingly, we discover that the solution obtained, that is equation (2.57) has a particular link to the pure Newtonian gravitational scalar potential for the gravitational field and hence put Einstein's geometrical gravitational field on the same level with the Newtonian dynamical theory of gravitation as obtained by [4,6,19].

The gravitational scalar potential obtained in this research work can be applied in

- the study of rotating astrophysical bodies within a spherical geometry whose tensor field varies with time, radial distance and polar angle. Example of such bodies are stars such as Neutron star, Wolf-Rayet, e.t.c.
- the study of astrophysical phenomenon such as gravitational red-shift by the sun, time dilation, length contraction, motion of particles and photons
- the study of gravitomagnetic and gravitoelectric coupling, just to mention but a few.

Thus we have completely obtained the solution of EGGFE exterior to a homogeneous spherical bodies whose tensor fields varies with time, radial distance and polar angle.

The solutions of the Einstein's field equations are metrics of space time. These metrics describe the structure of the space-time including the inertial motion of objects in the space-time. As the field equations are non-linear, they cannot always be completely solved (that is without approximation).

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