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The Resemblance between the Christoffel Symbols Derived from the Howusu Metric Tensor and That of the Schwarzschild Metric Tensor

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Abstract
A new set of Christoffel symbols $\Gamma_{\alpha\beta}^{\delta}$ were derived using the Howusu metric tensor. Results obtained were compared with the well-known Christoffel symbols $\Gamma_{\alpha\beta}^{\delta}$ derived from the Schwarzschild metric tensor by varying the radial distance. It was found that, at r = 0, the Howusu metric tensor behaved slightly different from the Schwarzschild metric tensor, but behaved exactly alike at r = ∞.

Keywords : Christoffel Symbols; Howusu Metric Tensor; Schwarschild Metric Tensor.

Introduction

After Albert Einstein Published his Einstein Field equation in 1915, the exact and analytical solutions for all the gravitational fields in nature were not found (Hoyle, 1948). However, Karl Schwarzschild introduced a metric tensor in 1916 called the Schwarzschild metric tensor for all gravitational fields around a symmetrically spherical object without angular momentum (Howusu, 2007; Howusu, 2010). This Schwarzschild metric can be summarized as

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{2M}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \quad (1)$$

Where M is the mass of the object and r is the distance away from the object. Then, S.K.X Howusu introduced a new metric tensor in 2009 called Howusu metric tensor for all gravitational fields in nature (Howusu, 2012). This metric is written as follows:

$$g_{\mu\nu} = \begin{pmatrix} -exp\left(\frac{2GM}{c^2r}\right) & 0 & 0 & 0 \\ 0 & exp\left(-\frac{2GM}{c^2r}\right) & 0 & 0 \\ 0 & 0 & r^2 exp\left(-\frac{2GM}{c^2r}\right) & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta\, exp\left(-\frac{2GM}{c^2r}\right) \end{pmatrix} \quad (2)$$

Where C is the speed of light; G is the universal constant of gravitation; M is the mass of the object and r is the distance away from the object. This article compares the Christoffel Symbol obtained using the Schwarzschild metric and that of the Howusu Metric.

Mathematical Theory

The Christoffel symbols are an array of numbers describing a metric connection with the following formula (Howusu & Uduh, 2003).

$$g_{\alpha\delta}\Gamma_{\beta\gamma}^{\delta} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right) \quad (3)$$

There were a total of 64 Christoffel symbols Γ because λ, μ, and ν each represented four components of the matrix: t, r, θ and Φ. So, the full set of the Christoffel Symbols must contain a combination of all the possibilities of these coordinates. However, one interesting property of Christoffel Symbols is (Kumar, 2009).

$$\Gamma_{\beta\gamma}^{\delta} = \Gamma_{\gamma\beta}^{\delta} \quad (4)$$

Setting, α=0, β=1, and γ=0, then substituting these values into equation (3), we got

$$g_{0\delta}\Gamma_{10}^{\delta} = \frac{1}{2}\left(\frac{\partial g_{01}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^1}\right) \quad (5)$$

Substituting equation (2) into equation (5), we arrived at

$$g_{0\delta}\Gamma_{10}^{\delta} = \frac{1}{2}\left(\frac{\partial(0)}{\partial t} + \frac{\partial\left(-exp\left(\frac{2GM}{c^2r}\right)\right)}{\partial r} - \frac{\partial(0)}{\partial r}\right) \quad (6)$$

Differentiating equation (6), we obtained

$$g_{0\delta}\Gamma_{10}^{\delta} = \frac{1}{2}\left(-\frac{2GM}{c^2r^2}exp\left(\frac{2GM}{c^2r}\right)\right) \quad (7)$$

$$g_{0\delta}\Gamma_{10}^{\delta} = -\frac{GM}{c^2r^2}exp\left(\frac{2GM}{c^2r}\right) \quad (8)$$

Summing over all the values of δ

$$g_{0\delta}\Gamma_{10}^{\delta} = \Gamma_{10}^{0} + g_{01}\Gamma_{10}^{1} + g_{02}\Gamma_{10}^{2} + g_{03}\Gamma_{10}^{3} \quad (9)$$

Referring to (2), the values g_{01}, g_{02} and g_{03} are all zero except for g_{00}

$$g_{0\delta}\Gamma^{\delta}_{10} = g_{00}\Gamma^{0}_{10} + 0 + 0 + 0 \quad (10)$$

$$g_{00}\Gamma^{0}_{10} = -\frac{GM}{c^2r^2}\, exp\left(\frac{2GM}{c^2r}\right) \quad (11)$$

Multiply both sides by

$$g^{00} = -exp\left(-\frac{2GM}{c^2r}\right)$$

$$\Gamma^{0}_{10} = \Gamma^{0}_{01} = \frac{GM}{c^2r^2} \quad (12)$$

Similarly, the mathematical results for the other non-zero terms from the calculation of the Christoffel symbols using the Howusu Metric Tensor were:

$$\Gamma_{00}{}^{1} = -\frac{GM}{c^2r^2} \quad (13)$$

$$\Gamma^{1}_{11} = -\frac{GM}{c^2r^2} \quad (14)$$

$$\Gamma^{1}_{22} = \frac{GM}{c^2} - r \quad (15)$$

$$\Gamma^{1}_{33} = \frac{GM}{c^2}sin^2\theta - rsin^2\theta \quad (16)$$

$$\Gamma^{2}_{21} = \Gamma^{2}_{12} = \frac{1}{r} - \frac{GM}{c^2r^2} \quad (17)$$

$$\Gamma^{2}_{33} = -\cos\theta\sin\theta \quad (18)$$

$$\Gamma^{3}_{13} = \Gamma^{3}_{31} = \frac{1}{r} - \frac{GM}{c^2r^2} \quad (19)$$

$$\Gamma^{3}_{23} = \Gamma^{3}_{32} = \frac{\cos\theta}{\sin\theta} = \cot\theta \quad (20)$$

Where c is the speed of light; G is the universal constant of gravitation; M is the mass of the object and r is the distance away from the object.

Comparing (12)–(20) with the well-known Christoffel symbols calculated from the Schwarzschild Metric Tensor.

Results and Discussion

It can be seen that in the case of the Howusu Metric Tensor, the Christoffel symbols Γ^0_{10} and Γ^0_{01} were equal quantitatively from (12). Clearly, as r→0, both Γ^0_{10} and Γ^0_{01} tended to infinity. On the other hand, both Γ^0_{10} and Γ^0_{01} tended to zero as r→∞.

For the Schwarzschild Metric Tensor, the Christoffel symbol Γ^0_{10} and Γ^0_{01} were also equal quantitatively and as r→ 0 Γ^0_{10} and Γ^0_{01} = - ∞. On the other hand, Γ^0_{10} and Γ^0_{01} = 0 as r→∞.

Taking the Howusu Metric Tensor, the Christoffel symbols Γ^1_{00} in (13)
It can be seen that as r→0, Γ^1_{00}=∞. Going further, as r→∞, Γ^1_{00} tended to zero.In the case of the Schwarzschild Metric Tensor, the Christoffel symbol Γ^1_{00}, as r→0, Γ^1_{00} =-∞. Going further as r→∞, Γ^1_{00} tended to zero.

Consider the Howusu Metric Tensor, the Christoffel symbol Γ^1_{11} given in (14)

It can be seen that as r→0, Γ^1_{11}= - ∞.Going further, as r→∞, Γ^1_{11} tended to zero. In the case of the Schwarzschild Metric Tensor, the Christoffel symbol Γ^1_{11} was
Clearly seen that, as r→0, Γ^1_{11}= ∞. Going further, as r→∞, Γ^1_{11} tended to zero.

The Christoffel symbol Γ^1_{22} in the Howusu Metric Tensor was GM/c^2 , as r tended to zero. Going further, as r→∞, Γ^1_{22} tended to negative infinity. For the Schwarzschild Metric Tensor, the Christoffel symbol Γ^1_{22} it was clear that, as r→0,
Γ^1_{22} = 2M. Going further, as r→∞, Γ^1_{22} tended to negative infinity.
In the case of the Howusu Metric Tensor, the Christoffel Symbols Γ^1_{33} in (16)

It can be seen that as r tended to zero, Γ^1_{33}=GM/c^2 $\sin^2\theta$ and as r→∞, Γ^1_{33} tended to negative infinity. On the other hand, in the Christoffel symbol Γ^1_{33} for the Schwarzschild Metric Tensor, as r→0, Γ^1_{33}=2M $\sin^2\theta$ and, as r→∞, Γ^1_{33} tended to negative infinity.

Taking the Howusu Metric, the Christoffel symbols Γ^2_{22} and Γ^2_{21} were equal quantitatively, according to (17)

Clearly, as r→0, both Γ^2_{12} and Γ^2_{21} tended to infinity. On the other hand, both Γ^2_{12} and Γ^2_{21} tended to zero as r→∞.For the Schwarzschild metric, the Chirstoffel Symbols Γ^2_{12} and Γ^2_{21} were equal quantatively. Clearly, as r→0, Γ^2_{12} and Γ^2_{21} tended to infinity, and Γ^2_{12} and Γ^2_{21} as r→∞.

In the case of the Howusu Metric Tensor, the Christoffel symbol Γ^2_{33}, (18) showed that the radial coordinate was unaffected whenever there were changes in the r coordinate, where θ is the angular coordinate of the system. Likewise, for the Schwarzschild Metric Tensor, the Christoffel symbol Γ^2_{33} also showed that the radial coordinate was unaffected whenever there was a change in the coordinate.

In the case of the Howusu Metric Tensor, the Christoffel symbols Γ^3_{13} and Γ^3_{31} were equal quantitatively from (19). Clearly, as r→0, both Γ^3_{13} and Γ^3_{31} are zero while both Γ^3_{13} and Γ^3_{31} tended to be zero as r→∞. For the Schwarzschild metric, the Christoffel Symbol Γ^3_{13} and Γ^3_{31} were equal quantitatively, it can be seen that as r→0, Γ^3_{13} and Γ^3_{31} tended to infinity. The quantities Γ^3_{13}=Γ^3_{31}=0 as r→∞.

For the Howusu Metric Tensor, the Christoffel symbol Γ^3_{23}, (20), the radial coordinate was shown to be unaffected whenever there was a change in the coordinate. Where θ is the angular coordinate of the system. Likewise, for the Schwarzschild Metric Tensor, the Christoffel Symbol Γ^3_{23}, the radial coordinate was unaffected whenever there is a change in the coordinate.

Conclusion

The Schwarzschild Metric Tensor and the Howusu Metric Tensor were investigated. The Christoffel Symbols ($\Gamma_{\beta\gamma}{}^{\delta}$)

derived from the Howusu Metric Tensor were compared to the Christoffel Symbol derived from the Schwarzschild Metric Tensor. Results of the analysis indicated that the behavior of the Howusu $\Gamma_{\beta\gamma}^{\delta}$ was, slightly different from the behavior of the Schwarzschild $\Gamma_{\beta\gamma}^{\delta}$ in the limit as the radial distance $r \to 0$. Similar results were also obtained in the limit as $r \to \infty$. The Christoffel Symbols (5) - (13) derived in this paper have paved the way for the comparison of Howusu's Christoffel symbols with Christoffel symbols derived from another known metric tensor.

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