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Derivation of Laplacian Equation from Exterior Einstein Geometrical Field Equation Using Golden Metric Tensor Approach in Weak Field Limit

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Abstract
The laplacian equation is a second –order partial differential equation which is useful for the determination of the electric potential in free space or region. In this article, the Riemannian geometry of space-time was applied to obtain affine connection coefficients, Riemann christofell tensor, Ricci tensor and exterior Einstein's field equation for spherical field. The result obtained in the limit of weak field reduces to laplacian equation which agrees with the concept of general relativity, and has a gravitational scalar potential of two functions, which does not differ significantly from Newton dynamical theory of gravitation. The solution further confirms the assumption that Newton dynamical theory of gravitation is a limiting case of Einstein's geometrical gravitational theory of gravitation.

Keywords : Riemannian geometry, laplacian equation, golden metric tensor, gravitational scalar potential.

Introduction

Einstein 's theory of gravity, is the geometric theory of gravitation published by Albert Einstein in 1915 and the current description of the gravitation in modern Physics (Bergmann, 1947). General relativity generalizes special relativity and refines Newton law of universal gravitation, providing a unified description of gravity as a geometric property of space and time or four dimensional space-time (Weinberg, 1972).

The term General Relativity is the most widely accepted theory of gravitation (Howusu, 2010; Chifu, 2012). The equations are in the form of tensor equation which related the local space-time curvature expressed by the Einstein tensor with the local energy and momentum within that space-time expressed by the stress-energy tensor (Misner et al., 1973). In this article from the Einstein geometrical field equations for homogenous spherical bodies with tensor field that varies with time and radial distance using Riemannian golden metric tensor we obtained laplacian equation in the weak field limit and is second –order partial differential equation which consists of two important properties. The first property is that the solution of laplacian equation is unique once solved under suitable number of boundary condition used and second property is that the solution of laplacian equation hold good with the superposition principle.

The Laplacian occurs in many differential equations describing physical phenomena. The general theory of solution to Laplacian equation is known as potential theory.

Construction of Metric Tensors and Affine Connections

Consider a body in spherical geometry with a tensor field that varies with time and radial distance. The coefficient of affine connection were calculated (Schwarzschild's, 1916; Howusu, 2008; Chifu & Howusu, 2008) using the equation below:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\xi}\left(g_{\alpha\xi\beta} + g_{\alpha\xi\beta} - g_{\alpha\beta\xi}\right) \quad (1)$$

Where,

$\Gamma_{\alpha\beta}{}^{\mu}$ = coefficient of affine connection

$g^{\mu\xi}$ = covariant metric tensor

$g_{\alpha\xi\beta}$ = contravariant metric tensor

The covariant metric tensors for this distribution of mass or pressure is given by (Howusu, 2009; Howusu, 2007).

$$g_{00} = -\left[1 + \frac{2}{c^2} f(t,r)\right] \quad (2)$$

$$g_{11} = \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \quad (3)$$

$$g_{22} = r^2\left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \quad (4)$$

$$g_{33} = r^2 \sin^2\theta\left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \quad (5)$$

$$g_{\mu\nu} = 0, \text{ Otherwise} \quad (6)$$

Where,

$f(t, r)$ is a gravitational scalar potential, determined by the mass or pressure and possess symmetries of the latter's. In approximate gravitational field, it is equal to Newton's gravitational scalar potential exterior to the spherical mass distribution.

The contravariant metric tensors in spherical polar coordinate in the are given by (Gupta, 2010)

$$g^{00} = -[1+\frac{2}{c^2}f(t,r)]^{-1} \tag{7}$$

$$g^{11} = [1+\frac{2}{c^2}f(t,r)] \tag{8}$$

$$g^{22} = \frac{1}{r^2}[1+\frac{2}{c^2}f(t,r)] \tag{9}$$

$$g^{33} = \frac{1}{r^2\sin^2\theta}[1+\frac{2}{c^2}f(t,r)] \tag{10}$$

$$g^{\mu\nu} = 0, \text{ Otherwise} \tag{11}$$

To obtain the coefficient of affine connection, we use the covariant and contravariant metric tensors, the affine connection coefficient are given by

$$\Gamma^0_{00} = \frac{1}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\frac{\partial f}{\partial t} \tag{12}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\frac{\partial f}{\partial r} \tag{13}$$

$$\Gamma^0_{11} = -\frac{1}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-3}\frac{\partial f}{\partial t} \tag{14}$$

$$\Gamma^0_{22} = -\frac{r^2}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-3}\frac{\partial f}{\partial t} \tag{15}$$

$$\Gamma^0_{33} = -\frac{r^2\sin^2\theta}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-3}\frac{\partial f}{\partial t} \tag{16}$$

$$\Gamma^1_{00} = \frac{1}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)\frac{\partial f}{\partial r} \tag{17}$$

$$\Gamma^1_{01} = \Gamma^1_{10} = -\frac{1}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\frac{\partial f}{\partial t} \tag{18}$$

$$\Gamma^1_{11} = -\frac{1}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\frac{\partial f}{\partial r} \tag{19}$$

$$\Gamma^1_{22} = -r+\frac{r^2}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\frac{\partial f}{\partial r} \tag{20}$$

$$\Gamma^2_{02} = \Gamma^2_{20} = -\frac{1}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\frac{\partial f}{\partial t} \tag{21}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}-\frac{1}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\frac{\partial f}{\partial r} \tag{22}$$

$$\Gamma^3_{03} = \Gamma^3_{30} = -\frac{1}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\frac{\partial f}{\partial t} \tag{23}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}-\frac{1}{c^2}\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\frac{\partial f}{\partial r} \tag{24}$$

$$\Gamma^{\mu}_{\alpha\beta} = 0; \text{ Otherwise} \tag{25}$$

Construction of Einstein equation

The Einstein's field equation (EFE) exterior to a homogeneous spherical distribution of mass is given by (Misner et al., 1973; Tajmar, 2001; Howusu, 2008; Chifu, & Howusu, 2009).

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{26}$$

where,

G_{uv} = Einstein's Tensors

R_{uv} = *Ricci Tensors*

R = Riemann Scalar

g_{uv} = Covariant Metric Tensor

It is observed (Misner *et al.,* 1973) that the exterior field equations along the G_{11}, G_{22} and G_{33} converge within the exterior field, similarly along the interior field.

For mathematical convenience, we choose G_{00}

Hence the field equation is given by

$$G_{00} = R_{00} - \frac{1}{2}Rg_{00} = 0 \tag{27}$$

The coefficient of affine connections of this field were used to construct the Ricci tensor and the curvature scalar given respectively as:

$$R_{00} = \frac{12}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 - \frac{3}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\frac{\partial^2 f(t,r)}{\partial t^2} - \frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial^2 f(t,r)}{\partial r^2} - \frac{2}{c^2 r}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial f(t,r)}{\partial r} + \frac{2}{c^4}\left(\frac{\partial f(t,r)}{\partial r}\right)^2 \tag{28}$$

$$R = -\frac{30}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 + \frac{6}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\frac{\partial^2 f(t,r)}{\partial t^2} + \frac{4}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 - \frac{2}{c^2}\frac{\partial^2 f(t,r)}{\partial r^2} - \frac{4}{c^2 r}\frac{\partial f(t,r)}{\partial r} + \frac{2}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)^2 + \frac{2}{r^2}\left[1+\frac{2f(t,r)}{c^2}\right] \tag{29}$$

Thus Substituting equation (28) (29) and (2) into (27) equation and rearranging gives

$$G_{00} = -\frac{2}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial^2 f(t,r)}{\partial r^2} - \frac{4}{c^2 r}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial f(t,r)}{\partial r} + \frac{2}{c^4}\left(\frac{\partial f(t,r)}{\partial r}\right)^2 + \frac{1}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial r}\right)^2 + \frac{2}{c^4}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 - \frac{3}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 + \frac{1}{r^2}\left[1+\frac{2f(t,r)}{c^2}\right]^2 = 0 \tag{29}$$

Multiply (29) through by $-\frac{2}{c^2}$ and dividing through by $\left[1+\frac{2f(t,r)}{c^2}\right]$ yields

$$\frac{\partial^2 f(t,r)}{\partial r^2} + \frac{2}{r}\frac{\partial f(t,r)}{\partial r} - \frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial r}\right)^2 - \frac{1}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)^2 - \frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 + \frac{3}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 - \frac{c^2}{2r^2}\left[1+\frac{2f(t,r)}{c^2}\right] = 0 \tag{30}$$

In the weak field limit of the order c0, equation (30) reduces to

$$\nabla^2 f(t,r) = 0 \qquad (31)$$

Equation (31) is known as Laplacian equation and ∇^2 is known as the Laplacian operator and $f(t,r)$ is a gravitational scalar potential.

Conclusion

From the result obtained in equation (31), we have established the fact that for a weak gravitational field, the exterior Einstein's geometrical gravitational field equation reduces Laplacian equation in the weak field limit of the order c0 which does not differ significantly from Newton dynamical theory of gravitation. But for intense gravitational field, the result does not reduce to Laplacian equation and diverges from that of Newton's gravitational theory because of additional correctional terms which are not found in the existing once.

Interestingly, we also discover that the solution obtained, that is equation (31) is the Newton dynamical scalar field equation. It is indeed a profound discovery, it confirms the assumption made by (Sarki et al., 2018); that Newton dynamical theory of gravitation (NDTG) is a limiting case of Einstein's geometrical gravitational field equations (EGGFE). It Experimentally shows equivalence principle of physics with the dependency of the gravitational scalar function on time and radial distance only.

The laplacian equation obtained in this research work can find application in the following field

- Any equation which is directly related to a linear differential equation can be easily solved using laplacian equation.
- The laplacian equation are used to describe the steady-state conduction heat transfer without any heat sources or sink.
- Laplacian equation can be used to determine the potential at any point between two surfaces when the potential of both surfaces is known.
- The capacitance between two surfaces can be found using Laplace's and Poisson's equation

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