Research Article

International Journal of Mathematic Exploration and Computer Education

Basis-Invariant Derivation of Numerical Discrete Schemes via Interpolation, Collocation and Evaluation at Identical Points

Subair Ahmed Olajuwon (Ph.D)

Nigerian Defence Academy Kaduna, Nigeria.

***Correspondence author**
Subair Ahmed Olajuwon (Ph.D)
Nigerian Defence Academy Kaduna,
Nigeria.

Submitted : 1 Oct 2025 ; Published : 15 Nov 2025

Citation: Subair, A.O. (2025). Basis-Invariant Derivation of Numerical Discrete Schemes via Interpolation, Collocation and Evaluation at Identical Points. *Int J Math Expl & Comp Edu.2*(3):1-4.

Abstract

This paper investigates the effect of basis selection on the derivation of numerical discrete schemes. In many numerical methods, especially those derived via interpolation and collocation techniques, such as Trigonometric Series, Orthogonal Functions, Polynomials and Power series, the choice of basis function is often assumed to influence the resulting discrete formulation. However, this study establishes that, irrespective of the kind of basis employed, the resulting numerical discrete scheme remains invariant provided that interpolation, collocation, and evaluation are performed at the same set of points. The invariance underscores the fundamental role of the interpolation and collocation nodes, rather than the basis itself, in determining the final scheme. The findings offer a unified perspective on scheme construction, reducing computational redundancy and strengthening the theoretical understanding of discrete approximations in numerical analysis.

Keywords: Numerical Discrete Scheme, Interpolaton, Collocation point, Basis-Invaroant, evaluation Point.

Introduction

In numerical analysis, discrete schemes derived from a general pth-order linear multistep method of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^p \sum_{j=0}^{k} \beta_j f_{n+j} \qquad j = 0,1,2,\dots k \qquad (1)$$

α_j and β_j are constants, y_{n+j} and f_{n+j} are interpolation and collocation points respectively, and p is the order of the method, are widely employed to approximate the solutions of differential equations, integral equations, and other functional problems. These schemes are typically derived using interpolation, collocation, and evaluation techniques, which collectively determine the coefficients and accuracy of the resulting numerical method. A common perception among researchers is that the choice of basis-functions such as an infinite sum of monomials known as power series used by Badmus and Yahaya (2009), Badmus et al. (2014), Blessing et al (2024), orthogonal polynomials by Badmus and Subair (2024) or trigonometric functions like that of Tian et al (2018) and Navnit and Kritika (2023), significantly influences the formulation and performance of the numerical discrete scheme. Consequently, much effort is often devoted to deriving multiple schemes from different bases in pursuit of enhanced accuracy or stability.

However, when interpolation, collocation, and evaluation are performed at the same set of points, theoretical arguments suggest that the resulting numerical discrete schemes should be invariant to the choice of basis. This observation implies that the interpolation nodes, collocation points, and evaluation criteria exert a more fundamental influence on the scheme than the bases functions themselves. Despite this intuitive expectation, a unified and rigorous demonstration of this invariance has remained limited in the existing literature.

This paper addresses this gap by establishing the basis-invariance of numerical discrete schemes derived under consistent interpolation, collocation, and evaluation points. By developing a generalized framework, we show that the coefficients and numerical properties of such schemes are essentially determined by the choice of nodes rather than by the selected basis functions. The findings simplify the derivation process, reduce computational redundancy, and provide a clearer theoretical understanding of discrete approximations in numerical analysis.

Definition of Terms

1. **Interpolation Point:** A point at which the solution of a function is evaluated.
2. **Collocation Point:** A point at which the derivative of a function is evaluated.
3. **Numerical Discrete Scheme:** This is an algorithmic description which requires iterative techniques to approximate solutions to mathematical equations of any order of derivatives, by employing values from neighboring points in a discretized n-dimensional grid to achieve convergence in solving equations.

Methodology

We compare two linear multistep method's discrete schemes from different bases functions. The first, Badmus et al (2014) at $k = 3$ with a power series basis, and the second a newly discrete schemes with Legendre polynomial as the basis under the same identical interpolation, collocation and evaluation conditions for fairness. The former derived a resulting discrete schemes expressed as

$$
\begin{aligned}
y_n &= -\frac{225}{128}y_{n+\frac{1}{2}} + \frac{25}{16}y_{n+\frac{3}{2}} + \frac{153}{128}y_{n+\frac{5}{2}} - \frac{225}{128}hf_{n+\frac{1}{2}} - \frac{75}{32}hf_{n+\frac{3}{2}} - \frac{45}{128}hf_{n+\frac{5}{2}} \\
y_{n+1} &= \frac{45}{128}y_{n+\frac{1}{2}} + \frac{9}{16}y_{n+\frac{3}{2}} + \frac{11}{128}y_{n+\frac{5}{2}} + \frac{9}{128}hf_{n+\frac{1}{2}} - \frac{9}{32}hf_{n+\frac{3}{2}} - \frac{3}{128}hf_{n+\frac{5}{2}} \\
y_{n+2} &= \frac{11}{128}y_{n+\frac{1}{2}} + \frac{9}{16}y_{n+\frac{3}{2}} + \frac{45}{128}y_{n+\frac{5}{2}} + \frac{3}{128}hf_{n+\frac{1}{2}} + \frac{9}{32}hf_{n+\frac{3}{2}} - \frac{9}{128}hf_{n+\frac{5}{2}} \\
y'_n &= \frac{1}{64h}\left(915y_{n+\frac{1}{2}} - 480y_{n+\frac{3}{2}} - 435y_{n+\frac{5}{2}} + 465hf_{n+\frac{1}{2}} + 820hf_{n+\frac{3}{2}} + 129hf_{n+\frac{5}{2}}\right) \\
y'_{n+2} &= \frac{1}{64h}\left(3y_{n+\frac{1}{2}} - 96y_{n+\frac{3}{2}} + 93y_{n+\frac{5}{2}} + hf_{n+\frac{1}{2}} - 12hf_{n+\frac{3}{2}} - 15f_{n+\frac{5}{2}}\right)
\end{aligned}
\qquad (2)
$$

Selection of Nodes using Legendre Polynomial as basis

Consider the linear multistep method of the form

$$y(x) = \sum_{j=0}^{s+t-1} \alpha_j P_j(x) = y_{n+j} \qquad (3)$$

and

$$y(x) = \sum_{j=0}^{s+t-1} \alpha_j P_j(x) = y_{n+j} \qquad (4)$$

where $(s + t - 1)$ is the degree of the polynomial and $P_j(x)$ a legendre polynomial with the first few terms expressed as

$$
\begin{gathered}
1 \\
x_n \\
\frac{1}{2}(3x_n^2 - 1) \\
\frac{1}{2}(5x_n^3 - 3x_n) \\
\frac{1}{8}(35x_n^4 - 30x_n^2 + 3) \\
\frac{1}{8}(63x_n^5 - 70x_n^3 + 15x_n)
\end{gathered}
\qquad (5)
$$

Interpolate (3) and collocate (4) at $x = \left(x_{n+\frac{1}{2}}, x_{n+\frac{3}{2}}, x_{n+\frac{5}{2}}\right)$ to form the d-matrix as follows:

$$
D = \begin{bmatrix}
1 & x_{n+\frac{1}{2}} & \frac{\left(3x^2_{n+\frac{1}{2}}-1\right)}{2} & \frac{\left(5x^3_{n+\frac{1}{2}}-3x_{n+\frac{1}{2}}\right)}{2} & \frac{\left(35x^4_{n+\frac{1}{2}}-30x^2_{n+\frac{1}{2}}+3\right)}{8} & \frac{\left(63x^5_{n+\frac{1}{2}}-70x^3_{n+\frac{1}{2}}+15x_{n+\frac{1}{2}}\right)}{8} \\
1 & x_{n+\frac{3}{2}} & \frac{\left(3x^2_{n+\frac{3}{2}}-1\right)}{2} & \frac{\left(5x^3_{n+\frac{3}{2}}-3x_{n+\frac{3}{2}}\right)}{2} & \frac{\left(35x^4_{n+\frac{3}{2}}-30x^2_{n+\frac{3}{2}}+3\right)}{8} & \frac{\left(63x^5_{n+\frac{3}{2}}-70x^3_{n+\frac{3}{2}}+15x_{n+\frac{3}{2}}\right)}{8} \\
1 & x_{n+\frac{5}{2}} & \frac{\left(3x^2_{n+\frac{5}{2}}-1\right)}{2} & \frac{\left(5x^3_{n+\frac{5}{2}}-3x_{n+\frac{5}{2}}\right)}{2} & \frac{\left(35x^4_{n+\frac{5}{2}}-30x^2_{n+\frac{5}{2}}+3\right)}{8} & \frac{\left(63x^5_{n+\frac{5}{2}}-70x^3_{n+\frac{5}{2}}+15x_{n+\frac{5}{2}}\right)}{8} \\
0 & 1 & 3x_{n+\frac{1}{2}} & \frac{\left(15x^2_{n+\frac{1}{2}}-3\right)}{2} & \frac{\left(140x^3_{n+\frac{1}{2}}-60x_{n+\frac{1}{2}}\right)}{8} & \frac{\left(315x^4_{n+\frac{1}{2}}-210x^3_{n+\frac{1}{2}}+15\right)}{8} \\
0 & 1 & 3x_{n+\frac{3}{2}} & \frac{\left(15x^2_{n+\frac{3}{2}}-3\right)}{2} & \frac{\left(140x^3_{n+\frac{3}{2}}-60x_{n+\frac{3}{2}}\right)}{8} & \frac{\left(315x^4_{n+\frac{3}{2}}-210x^3_{n+\frac{3}{2}}+15\right)}{8} \\
0 & 1 & 3x_{n+\frac{5}{2}} & \frac{\left(15x^2_{n+\frac{5}{2}}-3\right)}{2} & \frac{\left(140x^3_{n+\frac{5}{2}}-60x_{n+\frac{5}{2}}\right)}{8} & \frac{\left(315x^4_{n+\frac{5}{2}}-210x^3_{n+\frac{5}{2}}+15\right)}{8}
\end{bmatrix}
\begin{bmatrix} \alpha_{\frac{1}{2}} \\ \alpha_{\frac{3}{2}} \\ \alpha_{\frac{5}{2}} \\ \beta_{\frac{1}{2}} \\ \beta_{\frac{3}{2}} \\ \beta_{\frac{5}{2}} \end{bmatrix}
=
\begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{2}} \\ y_{n+\frac{5}{2}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{2}} \\ f_{n+\frac{3}{2}} \end{bmatrix}
\qquad (6)
$$

The continuous formulation and the continuous scheme of equation (6) are respectively given by

$$y(x) = \alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} + \alpha_{\frac{3}{2}} y_{n+\frac{3}{2}} + \alpha_{\frac{5}{2}} y_{n+\frac{5}{2}} = \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} \tag{7}$$

and

$$\begin{aligned}
y(x) = {} & \Bigg(-\frac{1}{1920h^5}\left(3375h^5 + 2740h^4x_n + 48840h^3x_n^2 + 35760h^2x_n^3 + 11760hx_n^4 + 1440x_n^5 + \right. \\
& \left. 16280h^3 + 35760h^2x_n + 23520hx_n^2 + 4800x_n^3 + 2352h + 1440x_n\right) + \frac{1}{2240h^5}\left([32025h^4 + \right. \\
& 113960h^3x_n + 125160h^2x_n^2 + 54880hx_n^3 + 8400x_n^4 + 25032h^2 + 32928hx_n + 10080x_n^2 + \\
& \left. 4720]x_n\right) - \frac{1}{168h^5}\left(2849h^3 + 6258h^2x_n + 4116hx_n^2 + 840x_n^3 + 588h + 360x_n\right)\left(\frac{3x_n^2}{2} - \frac{1}{2}\right) + \\
& \frac{1}{60h^5}\left(447h^2 + 588hx_n + 180x_n^2 + 20\right)\left(\frac{5x_n^3}{2} - \frac{3x_n}{2}\right) - \frac{1}{35h^5}\left(49h + 30x_n\right)\left(\frac{35x_n^4}{8} - \frac{15x_n^2}{4} + \frac{3}{8}\right) + \\
& \frac{2}{21h^5}\left(\frac{63x_n^5}{8} - \frac{35x_n^3}{4} + \frac{15x_n}{8}\right)\Bigg) y_{n+\frac{1}{2}} + \Bigg(\frac{1}{240h^4}\left(375h^4 + 1800h^3x_n + 2760h^2x_n^2 + 1440hx_n^3 + 240x_n^4 + \right. \\
& \left. 920h^2 + 1440hx_n + 480x_n^2 + 48\right) - \frac{1}{10h^4}\left(75h^3 + 230h^2x_n + 180hx_n^2 + 40x_n^3 + 36h + 24x_n\right)x_n + \\
& \frac{1}{21h^4}\left(161h^2 + 252hx_n + 84x_n^2 + 12\right)\left(\frac{3x_n^2}{2} - \frac{1}{2}\right) - \frac{4}{5h^4}\left(2x_n + 3h\right)\left(\frac{5x_n^3}{2} - \frac{3x_n}{2}\right) + \frac{8}{35h^4}\left(\frac{35x_n^4}{8} - \frac{15x_n^2}{4} + \right. \\
& \left. \frac{3}{8}\right)\Bigg) y_{n+\frac{3}{2}} + \Bigg(\frac{1}{192h^5}\left(2295h^5 + 13050h^4x_n + 26760h^3x_n^2 + 24240h^2x_n^3 + 9840hx_n^4 + 1440x_n^5 + \right. \\
& \left. 8920h^3 + 24240h^2x_n + 19680hx_n^2 + 4800x_n^3 + 1968h + 1440x_n\right) - \frac{1}{2240h^5}\left([15225h^4 + \right. \\
& 62440h^3x_n + 84840h^2x_n^2 + 45920hx_n^3 + 8400x_n^4 + 16968h^2 + 27552hx_n + 10080x_n^2 + \\
& \left. 720]x_n\right) + \frac{1}{168h^5}\left(1561h^3 + 4242h^2x_n + 3444hx_n^2 + 840x_n^3 + 492h + 360x_n\right)\left(\frac{3x_n^2}{2} - \frac{1}{2}\right) - \\
& \frac{1}{60h^5}\left(303h^2 + 492hx_n + 180x_n^2 + 20\right)\left(\frac{5x_n^3}{2} - \frac{3x_n}{2}\right) + \frac{1}{35h^5}\left(41h + 30x_n\right)\left(\frac{35x_n^4}{8} - \frac{15x_n^2}{4} + \frac{3}{8}\right) - \\
& \frac{2}{21h^5}\left(\frac{63x_n^5}{8} - \frac{35x_n^3}{4} + \frac{15x_n}{8}\right)\Bigg) y_{n+\frac{5}{2}} + \Bigg(\frac{-1}{1920h^4}\left(3375h^5 + 13950h^4x_n + 20040h^3x_n^2 + 13200h^2x_n^3 + \right. \\
& \left. 4080hx_n^4 + 480x_n^5 + 6680h^3 + 13200h^2x_n + 8160hx_n^2 + 1600x_n^3 + 816h + 480x_n\right) + \\
& \frac{1}{2240h^4}\left([16275h^4 + 46760h^3x_n + 46200h^2x_n^2 + 19040hx_n^3 + 2800x_n^4 + 9240h^2 + 11424hx_n + \right. \\
& \left. 3360x_n^2 + 240]x_n\right) - \frac{1}{168h^4}\left(1169h^3 + 2310h^2x_n + 1428hx_n^2 + 280x_n^3 + 204h + 120x_n\right)\left(\frac{3x_n^2}{2} - \right. \\
& \left. \frac{1}{2}\right) + \frac{1}{180h^4}\left(495h^2 + 612hx_n + 180x_n^2 + 20\right)\left(\frac{5x_n^3}{2} - \frac{3x_n}{2}\right) - \frac{1}{35h^5}\left(17h + 10x_n\right)\left(\frac{35x_n^4}{8} - \frac{15x_n^2}{4} + \frac{3}{8}\right) - \\
& \frac{2}{63h^5}\left(\frac{63x_n^5}{8} - \frac{35x_n^3}{4} + \frac{15x_n}{8}\right)\Bigg) f_{n+\frac{1}{2}} + \Bigg(\frac{-1}{96h^4}\left(225h^5 + 1230h^4x_n + 2376h^3x_n^2 + 1968h^2x_n^3 + 720hx_n^4 + \right. \\
& \left. 96x_n^5 + 792h^3 + 1968h^2x_n + 1440hx_n^2 + 320x_n^3 + 144h + 96x_n\right) + \frac{1}{560h^4}\left([7175h^4 + \right. \\
& \left. 27720h^3x_n + 34440h^2x_n^2 + 16800hx_n^3 + 2800x_n^4 + 6888h^2 + 10080hx_n + 3360x_n^2 + 240]x_n\right) - \\
& \frac{1}{42h^4}\left(693h^3 + 1722h^2x_n + 1260hx_n^2 + 280x_n^3 + 180h + 120x_n\right)\left(\frac{3x_n^2}{2} - \frac{1}{2}\right) + \frac{1}{45h^4}\left(369h^2 + \right. \\
& \left. 540hx_n + 180x_n^2 + 20\right)\left(\frac{5x_n^3}{2} - \frac{3x_n}{2}\right) - \frac{4}{7h^5}\left(2h + 3x_n\right)\left(\frac{35x_n^4}{8} - \frac{15x_n^2}{4} + \frac{3}{8}\right) + \frac{8}{63h^5}\left(\frac{63x_n^5}{8} - \frac{35x_n^3}{4} + \right. \\
& \left. \frac{15x_n}{8}\right)\Bigg) f_{n+\frac{3}{2}} + \Bigg(\frac{-1}{1920h^4}\left(675h^5 + 3870h^4x_n + 8040h^3x_n^2 + 7440h^2x_n^3 + 3120hx_n^4 + 480x_n^5 + \right. \\
& \left. 2680h^3 + 7440h^2x_n + 6240hx_n^2 + 1600x_n^3 + 624h + 480x_n\right) + \frac{1}{2240h^4}\left([4515h^4 + 18760h^3x_n + \right. \\
& \left. 26040h^2x_n^2 + 14560x_n^3 + 2800x_n^4 + 5208h^2 + 8736hx_n + 3360x_n^2 + 240]x_n\right) - \frac{1}{42h^4}\left(469h^3 + \right. \\
& \left. 1302h^2x_n + 1092hx_n^2 + 280x_n^3 + 156h + 120x_n\right)\left(\frac{3x_n^2}{2} - \frac{1}{2}\right) + \frac{1}{180h^4}\left(279h^2 + 468hx_n + 180x_n^2 + \right. \\
& \left. 20\right)\left(\frac{5x_n^3}{2} - \frac{3x_n}{2}\right) - \frac{4}{35h^5}\left(13h + 10x_n\right)\left(\frac{35x_n^4}{8} - \frac{15x_n^2}{4} + \frac{3}{8}\right) + \frac{2}{63h^5}\left(\frac{63x_n^5}{8} - \frac{35x_n^3}{4} + \frac{15x_n}{8}\right)\Bigg) f_{n+\frac{5}{2}}
\end{aligned} \tag{8}$$

Evaluating the continuous scheme above at points $x_n = (y_n, y_{n+1}, y_{n+2})$, its first derivative evaluated at $x_n = (f_n, f_{n+2})$ gives

$$y_n = -\frac{225}{128}y_{n+\frac{1}{2}} + \frac{25}{16}y_{n+\frac{3}{2}} + \frac{153}{128}y_{n+\frac{5}{2}} - \frac{225}{128}hf_{n+\frac{1}{2}} - \frac{75}{32}hf_{n+\frac{3}{2}} - \frac{45}{128}hf_{n+\frac{5}{2}}$$

$$y_{n+1} = \frac{45}{128}y_{n+\frac{1}{2}} + \frac{9}{16}y_{n+\frac{3}{2}} + \frac{11}{128}y_{n+\frac{5}{2}} + \frac{9}{128}hf_{n+\frac{1}{2}} - \frac{9}{32}hf_{n+\frac{3}{2}} - \frac{3}{128}hf_{n+\frac{5}{2}}$$

$$y_{n+2} = \frac{11}{128}y_{n+\frac{1}{2}} + \frac{9}{16}y_{n+\frac{3}{2}} + \frac{45}{128}y_{n+\frac{5}{2}} + \frac{3}{128}hf_{n+\frac{1}{2}} + \frac{9}{32}hf_{n+\frac{3}{2}} - \frac{9}{128}hf_{n+\frac{5}{2}}$$

$$y'_n = \frac{1}{64h}\left(915y_{n+\frac{1}{2}} - 480y_{n+\frac{3}{2}} - 435y_{n+\frac{5}{2}} + 465hf_{n+\frac{1}{2}} + 820hf_{n+\frac{3}{2}} + 129hf_{n+\frac{5}{2}}\right)$$

$$y'_{n+2} = \frac{1}{64h}\left(3y_{n+\frac{1}{2}} - 96y_{n+\frac{3}{2}} + 93y_{n+\frac{5}{2}} + hf_{n+\frac{1}{2}} - 12hf_{n+\frac{3}{2}} - 15f_{n+\frac{5}{2}}\right) \quad (9)$$

Discussion of Result

The derivation of the Linear Multi step Method (LMM) using both the power series basis and the Legendre polynomial basis produced identical discrete schemes when the same interpolation, collocation, and evaluation points were employed as seen by equation (2) and equation (9). This outcome demonstrates that the coefficients of the discrete schemes are determined primarily by the choice of nodes, rather than by the particular form of a basis function.

Because both bases lead to the same scheme, it is unnecessary to reproduce the full derivations of their properties in this publication. Instead, it is sufficient to state that the Legendre polynomial based derivation reproduced the same numerical discrete scheme already obtained from the power series basis under the same conditions of interpolation, collocation and evaluation points. This directly confirms the basis-invariance principle: as long as interpolation, collocation, and evaluation are performed at identical points, the resulting discrete numerical methods are equivalent, irrespective of the underlying basis functions. Therefore, the order of accuracy and stability properties would be found to be equivalent.

This result simplifies the process of deriving new numerical schemes. Rather than repeating derivations for multiple bases, researchers can focus on selecting optimal nodes for interpolation and collocation, confident that the underlying basis choice will not affect the final scheme.

Conclusion

This study has demonstrated that, irrespective of the bases functions employed, the derivation of linear multistep methods using identical interpolation, collocation, and evaluation points leads to the same discrete scheme. This confirms that the choice of nodes and not the choice of basis govern the properties of the resulting numerical method and simplifies the process of method development. So, it strengthens the theoretical understanding of discrete approximations in numerical analysis.

The findings of this work establish a unifying principle: irrespective of the basis, the discrete schemes derived under consistent conditions will remain invariant. This principle can guide future research on higher-order methods, adaptive step sizes, and other families of bases functions, promoting more efficient derivations and a deeper insight into the structure of numerical methods.

References

1. Badmus, A., & Yahaya, Y. A. (2009). An Accurate Uniform Order 6 Block Method for Direct Solution of General Second Order Ordinary Differential Equations. *The Pacific Journal of Science and Technology, 10*(2), pp248-254. https://www.researchgate.net/publication/237637672_An_Accurate_Uniform_Order_6_Block_Method_for_Direct_Solution_of_General_Second_Order_Ordinary_Differential_Equations
2. Badmus, A. M., Yahaya, Y. A., & Subair, A. O. (2014). Implicit Stomer Cowell k-step Hybrid Block Methods for Solution of First Order Ordinary Differential Equations (ODEs). *International Journal of Applied Mathematical Research, 3*(4), pp464-472. DOI: https://doi.org/10.14419/ijamr.v3i4.3403
3. Badmus, A. M., & Subair, A. O. (2024). Implicit Block and Runge- Kutta type Methods for Solution of Second-Order Ordinary Differential Equations. *Applied Mathematics and Computational Intelligence, 13*(1), pp109-127. DOI: https://doi.org/10.58915/amci.v13iNo.1.229
4. Tian, Z., Li, X., Fan, C. M., & Chen, C. S. (2018). The Method of Particular Solutions Using Trigonometric Basis Functions. *Journal of Computational and Applied Mathematics, 335*(4), pp20-32. DOI: https://doi.org/10.1016/j.cam.2017.11.028
5. Navnit, J. & Kritika. (2023). Trigonometry Basis Approximated Fuzzy Components and High- Resolution Scheme for Two-Point Boundary Value Problems. Palestine Journal of Mathematics, 12(IV), pp21-32. https://pjm.ppu.edu/paper/1525-trigonometry-basis-approximated-fuzzy-components-and-high-resolution-scheme-two-point.